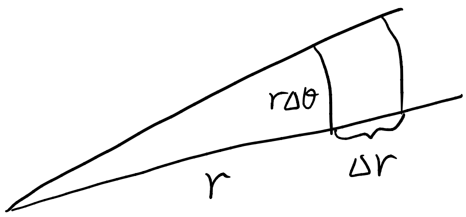
Polar Coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$



$\Delta V = r\Delta r\Delta\theta \Rightarrow dV = r\,dr\,d\theta$

Also,

$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$ $\quad \det J = r \Rightarrow dV = r\,dr\,d\theta$

Ex ① $\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$

$= 2\pi \int_0^{\infty} e^{-r^2}\,r\,dr$

$\overset{u=r^2}{=} 2\pi \int_0^{\infty} \frac{1}{2} e^{-u}\,du$

$= \pi$

② $\int_{\mathbb{R}} e^{-x^2}\,dx = ?$

Note $?^2 = \left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)\cdot\left(\int_{\mathbb{R}} e^{-y^2}\,dy\right)$

$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy$

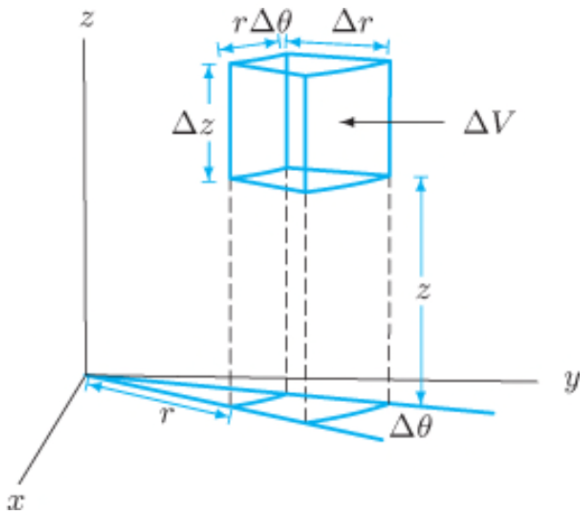
$= \pi$

$\Rightarrow ? = \sqrt{\pi}$

RELATION BETWEEN CARTESIAN AND CYLINDRICAL COORDINATES: Each point in $\mathbb{R}^3$ is represented using $0 \le r < \infty$, $0 \le \theta \le 2\pi$, $-\infty < z < \infty$.

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = z.$$

As with polar coordinates in the plane, note that $x^2 + y^2 = r^2$.



It is clear from this image that we should have $\Delta V \approx r\,\Delta r\,\Delta\theta\,\Delta z$. This leads us to the following conclusion: $dV = r\,dr\,d\theta\,dz$

Also, we can compute the Jacobian:

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det J = r \Rightarrow dV = r\,dr\,d\theta\,dz$$

Example 3.6.1. Find the volume of the solid region S which is above the half-cone given by $z = \sqrt{x^2 + y^2}$ and below the hemisphere where $z = \sqrt{8 - x^2 - y^2}$.

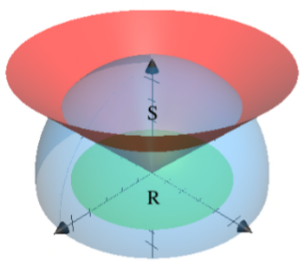


Figure 3.6.2: The "snow cone" S and the projected region R.

Solution:

Note that, in cylindrical coordinates, the half-cone is given by $z = \sqrt{r^2} = r$ and the hemisphere is given by $z = \sqrt{8 - r^2}$.

To find the volume, we need to calculate $\displaystyle\iiint_S dV$.

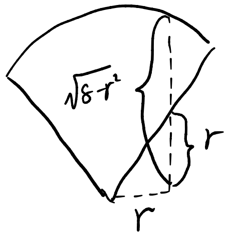
The projected region R in the xy-plane, or $r\theta$-plane, is the inside of the circle (thought of as lying in a copy of the xy-plane) along which the two surfaces intersect. To find this circle, we set the two z's equal to each other and find

$$r = \sqrt{8 - r^2}, \qquad \text{or, equivalently,} \qquad r^2 = 8 - r^2.$$

We find

$$2r^2 = 8, \qquad \text{so} \qquad r^2 = 4, \qquad \text{and, hence,} \qquad r = 2.$$

Thus, R is the disk in the xy-plane where $r \le 2$.



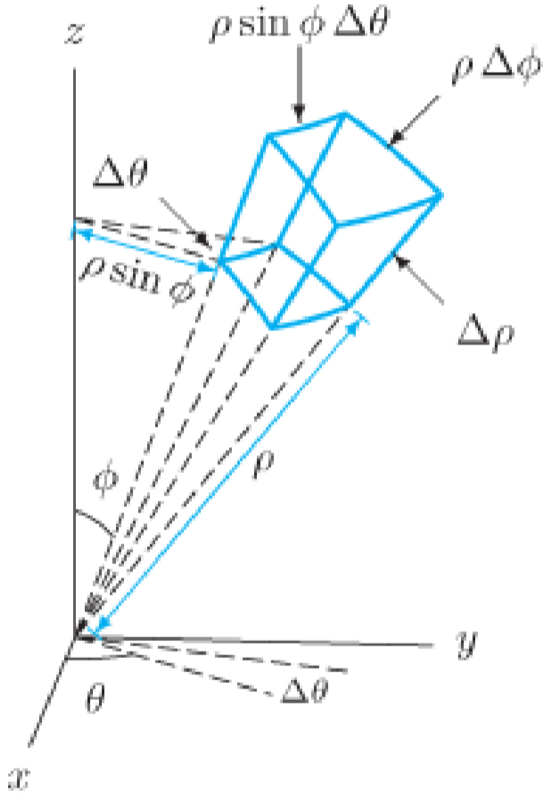
$$\iiint_S dV = \int_0^{2\pi}\int_0^{2}\int_r^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}\left(rz \Big|_{z=r}^{z=\sqrt{8-r^2}} \right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \left(r\sqrt{8-r^2} - r^2 \right) dr\,d\theta = \int_0^{2\pi} \frac{16}{3}(\sqrt{2}-1)\,d\theta = \frac{32}{3}(\sqrt{2}-1)$$

RELATIONSHIP BETWEEN CARTESIAN AND SPHERICAL COORDINATES: Each point in $\mathbb{R}^3$ is represented using $0 \le \rho < \infty$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

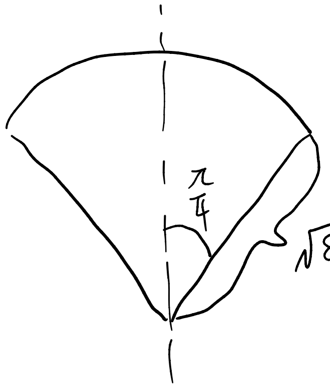
$$x = \rho\sin\phi\cos\theta,$$
$$y = \rho\sin\phi\sin\theta,$$
$$z = \rho\cos\phi.$$

Also, $x^2 + y^2 + z^2 = \rho^2$.



We can see that the small volume ΔV is approximated by $\Delta V \approx \rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$. This brings us to the conclusion about the volume element dV in spherical coordinates: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Ex. The same question as before, but compute it by spherical coor.



$$\int_S dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{8\sqrt{8}}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi \, d\phi \, d\theta$$

$$= \frac{16\sqrt{2}}{3}\cdot 2\pi \cdot \left(-\frac{\sqrt{2}}{2}+1\right)$$

$$= \frac{32\pi}{3}\left(\sqrt{2}-1\right)$$

Hyper-spherical coordinates :

Let $P = (x_1, \cdots, x_n) \in \mathbb{R}^n$.

$x_1 = r\cos\phi_1$

$x_2 = r\sin\phi_1 \cos\phi_2$

$x_3 = r\sin\phi_1 \sin\phi_2 \cos\phi_3$

$\vdots$

$x_{n-1} = r\sin\phi_1 \cdots \sin\phi_{n-2}\cos\phi_{n-1}$

$x_n = r\sin\phi_1 \cdots \sin\phi_{n-2}\sin\phi_{n-1}$